Static bounds on program running time

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Partial and total correctness

$$\{P\} \quad C \quad \{Q\}$$

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Partial Correctness

$(P \wedge h) \rightarrow Q$

Partial and total correctness

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Total Correctness	**Partial Correctness**
$P \rightarrow (h \wedge Q)$	$(P \wedge h) \rightarrow Q$

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✓ semantic bugs	✓ semantic bugs

Partial and total correctness

$$\{P\} \quad C \quad \{Q\}$$

Total Correctness	**Partial Correctness**
$P \to (h \land Q)$	$(P \land h) \to Q$
✓ semantic bugs	✓ semantic bugs
✓ nontermination	✗ nontermination

$$\{\mathtt{X} = n\}$$

```
Total := 0
I := 1
while I <= X :
   Total := Total + I
```

$$\{\mathtt{Total} = \frac{n(n+1)}{2}\}$$

Partial and total correctness

$$\{P\} \quad C \quad \{Q\}$$

Total Correctness	**Partial Correctness**
$P \to (h \land Q)$	$(P \land h) \to Q$
✓ semantic bugs	✓ semantic bugs
✓ nontermination	✗ nontermination
✗ performance bugs	✗ performance bugs

$$\{\mathtt{X} = n\}$$

```
Total := 0
I := 1
while I <= 2000000000 :
   if (I <= X) :
      Total := Total + I
   I := I + 1
```

$$\{\mathtt{Total} = \frac{n(n+1)}{2}\}$$

Partial and total correctness

$$\{P\} \quad C \quad \{Q\}$$

Total Correctness	$>$	**Partial Correctness**
$P \to (h \wedge Q)$		$(P \wedge h) \to Q$
✓ semantic bugs		✓ semantic bugs
✓ nontermination		✗ nontermination
✗ performance bugs		✗ performance bugs

Partial Correctness $+$ **Time** $>$ **Total Correctness** [1]

Overview

SPEED: Precise and Efficient Static Estimation of Program Computational Complexity

[2]: Sumit Gulwani, Krishna K Mehra, and Trishul Chilimbi, Microsoft Research, 2009

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Goal: Automatically generate upper bounds on the running time of simple programs

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Requirements:

- Able to contain $+, \cdot, \min, \max$

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Requirements:

- Able to contain $+, \cdot, \min, \max$
- Precise (tight)
- Correct

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Also:

★ Built from linear constraints

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- Able to contain $+, \cdot, \min, \max$
- Precise (tight)
- Correct

Also:

★ Built from linear constraints (allows use of linear invariant generation tool)

★ Small enough search space

```
def f1(x0, y0, m, n) :
   x := x0; y := y0
   while (x < m) :
      if (y < n) :
         y++
      else :
         x++
```

```
def f1(x0, y0, m, n) :
   x := x0; y := y0
   while (x < m) :
      if (y < n) :
         y++
      else :
         x++
```

Loop iterations: $(m - x_0) + (n - y_0)$

```
def f1(x0, y0, m, n) :
   x := x0; y := y0
   while (x < m) :
      if (y < n) :
         y++
      else :
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```

Loop iterations: 0 if $x_0 \geq m$; otherwise, $m - x_0 + \max(0, n - y_0)$.
Upper bound: $\max(0, m - x_0) + \max(0, n - y_0)$.

```
def f1(x0, y0, m, n) :
   x := x0; y := y0; T := 0
   while (x < m) :
      if (y < n) :
         y++
      else :
         x++
      T++
```

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def f1(x0, y0, m, n) :
   x := x0; y := y0; T := 0
   while (x < m) :
      if (y < n) :
         y++
      else :
         x++
      T++ {$T + x_0 + y_0 = x + y$}
```

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def f1(x0, y0, m, n) :
   x := x0; y := y0; T := 0
   while (x < m) :
      if (y < n) :
         y++
      else :
         x++
      T++ ${T + x_0 + y_0 = x + y \wedge x \leq n \wedge y \leq \max(n, y_0)}$
          $\implies \{T \leq \max(0, m - x_0) + \max(0, n - y_0)\}$
```

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```
def f1(x0, y0, m, n) :
   x := x0; y := y0; c1 := 0; c2 := 0
   while (x < m) :
      if (y < n) :
         y++; c2++
      else :
         x++; c1++
```

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Upper bound: $\max(0, m - x_0) + \max(0, n - y_0)$.

```
def f1(x0, y0, m, n) :
   x := x0; y := y0; c1 := 0; c2 := 0
   while (x < m) :
      if (y < n) :
         y++; c2++ $\{y_0 + c_2 = y \land y \leq n\}$
                   $\implies \{c_2 \leq n - y_0\}$
      else :
         x++; c1++ $\{x_0 + c_1 = x \land x \leq m\}$
                   $\implies \{c_1 \leq m - x_0\}$
```

$$\{T = c_1 + c_2 \leq \max(0, m - x_0) + \max(0, n - y_0)\}$$

Loop iterations: 0 if $x_0 \geq m$; otherwise, $m - x_0 + \max(0, n - y_0)$.
Upper bound: $\max(0, m - x_0) + \max(0, n - y_0)$.

```
def f2(x0, y0, n) :
x := x0; y := y0
while (x < n) :
   if (x < y) :
      x++
   else :
      y++
```

```
def f2(x0, y0, n) :
x := x0; y := y0
while (x < n) :
   if (x < y) :
      x++
   else :
      y++
```

Loop iterations:

- 0 if $x_0 \geq n$
- $n - x_0$ if $x_0 < n \leq y_0$
- $(n - x_0) + (n - y_0)$ if $x_0, y_0 < n$

Upper bound: $\max(0, m - x_0) + \max(0, n - y_0)$.

```
def f2(x0, y0, n) :
x := x0; y := y0; c1 := 0; c2 := 0
while (x < n) :
   if (x < y) :
      x++; c1++ ${x_0 + c_1 = x \land x \leq n}$
                  $\implies \{c_1 \leq n - x_0\}$
   else :
      y++; c2++ ${y_0 + c_2 = y \land y \leq n}$
                  $\implies \{c_2 \leq n - y_0\}$
```

$$\{T = c_1 + c_2 \leq \max(0, n - x_0) + \max(0, n - y_0)\}$$

Loop iterations:

- 0 if $x_0 \geq n$
- $n - x_0$ if $x_0 < n \leq y_0$
- $(n - x_0) + (n - y_0)$ if $x_0, y_0 < n$

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Strategy

- Assign a loop counter for every back-edge in the program

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$$\begin{aligned} c_3 &\leq B_1 \\ c_1 &\leq B_2 \\ c_2 &\leq B_3 \\ c_3 &\leq B_4 \\ c_1 &\leq B_5 \end{aligned}$$

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$$\begin{aligned} c_3 &\leq B_1 \\ c_1 &\leq B_2 \\ c_2 &\leq B_3 \\ c_3 &\leq B_4 \\ c_1 &\leq B_5 \end{aligned}$$

$$\begin{aligned} T &= c_1 + c_2 + c_3 \\ &\leq \max(0, B_2, B_5) + \max(0, B_3) + \max(0, B_1, B_4) \end{aligned}$$

```
def f3(m, n) :
x := 0; y := 0
while (x < m) :
   if (y < n) :
      y++
   else :
      y = 0
      x++
```

```
def f3(m, n) :
x := 0; y := 0
while (x < m) :
   if (y < n) :
      y++
   else :
      y = 0
      x++
```

Loop iterations:

- 0 if $m \leq 0$
- m if $n \leq 0 < m$
- $m(n+1)$ if $0 < m, n$

Upper bound: $\max(m, 0) + (\max(m, 0) + 1) \cdot \max(n, 0)$.

```
def f3(m, n) :
x := 0; y := 0; c1 := 0; c2 := 0
while (x < m) :
   if (y < n) :
      y++; c2++
   else :
      y = 0
      x++; c1++; c2 = 0
```

Loop iterations:

- 0 if $m \leq 0$
- m if $n \leq 0 < m$
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Upper bound: $\max(m, 0) + (\max(m, 0) + 1) \cdot \max(n, 0)$.

```
def f3(m, n) :
x := 0; y := 0; c1 := 0; c2 := 0
while (x < m) :
   if (y < n) :
      y++; c2++ ${y = c_2 \wedge y \leq n}$
                $\implies {c_2 \leq n}$
   else :
      y = 0
      x++; c1++; c2 = 0 ${x = c_1 \wedge x \leq m}$
                          $\implies {c_1 \leq m}$
```

Loop iterations:

- 0 if $m \leq 0$
- m if $n \leq 0 < m$
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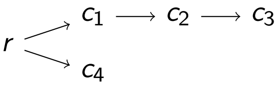
$$c_3 \leq B_1$$
$$c_1 \leq B_2$$
$$c_2 \leq B_3$$

$$c_4 \leq B_4$$
$$c_1 \leq B_5$$



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- Assign a loop counter for every back-edge in the program
- Assign DAG of loop counter dependencies
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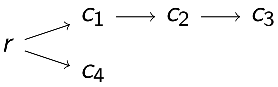
$$c_3 \leq B_1 \qquad c_4 \leq B_4$$
$$c_1 \leq B_2 \qquad c_1 \leq B_5$$
$$c_2 \leq B_3$$



$$T_1 \leq \max(0, B_2, B_5)$$
$$T_2 \leq (1 + T_1)\max(0, B_3)$$
$$T_3 \leq (1 + T_2)\max(0, B_1)$$
$$T_4 \leq \max(0, B_4)$$

$$T = T_1 + T_2 + T_3 + T_4$$

SPEED bound generation

Requirements:

- ✓ Able to contain $+, \cdot, \min, \max$
- ★ Precise (tight)
- ✓ Correct
- ✓ Built from linear constraints (allows use of linear invariant generation tool)
- ✓/★ Small enough search space

Search Space

- Pick a number of variables

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- Pick a number of variables
- Assign a variable to each back-edge ($>$ exponential)

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Optimal bound in search space

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- Assign a variable to each back-edge ($>$ exponential)
- Assign DAG of variable dependencies ($>$ exponential)

~~Optimal bound in search space~~

“Counter-optimal” bound

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- Assign a loop counter for every back-edge in the program
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And:

- Minimize number of counters

Strategy

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- Assign DAG of loop counter dependencies
- Compute a **linear** bound on the value of the loop counter at each back-edge

And:

- Minimize number of counters
- Minimize number of dependencies

- Minimize number of counters

```
def f4(n) :
   x = 0
   while (x < n) :
      if (*) :
         x++
      else
         x++
```

- Minimize number of counters

```
def f5(n) :
   x = 0
   while (x < n and rand({0,1}) == 0) :
      x++
   while (x < n) :
      x++
```

Algorithm

Repeat:

- Pick an unassigned back-edge and assign it a counter.

Algorithm

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- Pick an unassigned back-edge and assign it a counter.
    - If an existing counter works, use that.

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Repeat:

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    - If an existing counter works, use that.
    - Try to define a new counter.

Algorithm

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- Pick an unassigned back-edge and assign it a counter.
    - If an existing counter works, use that.
    - Try to define a new counter.
    - Otherwise, fail.

Algorithm

Repeat:

- Pick an unassigned back-edge and assign it a counter.
    - If an existing counter works, use that.
    - Try to define a new counter.
    - Otherwise, fail.

Whenever a new counter is added:

- Add all dependencies from previous counters.

Algorithm

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- Pick an unassigned back-edge and assign it a counter.
  - If an existing counter works, use that.
  - Try to define a new counter.
  - Otherwise, fail.

Whenever a new counter is added:

- Add all dependencies from previous counters.
- Remove one at a time until the invariant generation fails.

Implementation

- Quantitative functions on data structures

$$\text{len}\, A, \text{size}\, T, \text{location of } x \text{ in } L$$

Implementation

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- C/C++

Implementation

- Quantitative functions on data structures

$$\text{len } A, \text{size } T, \text{location of } x \text{ in } L$$

- C/C++
- Precise bounds on over 50% of loops in Microsoft product code

Possible research directions

- Nested max

$$\begin{aligned}&\max(0, m - x_0 + \max(0, n - y_0)) \\ &\leq \max(0, m - x_0) + \max(0, n - y_0)\end{aligned}$$

Possible research directions

- Nested max

$$\begin{aligned}&\max(0, m - x_0 + \max(0, n - y_0))\\ &\leq \max(0, m - x_0) + \max(0, n - y_0)\end{aligned}$$

- Optimal bound (rather than minimizing counters and dependencies)

Possible research directions

- Nested max

$$\max(0, m - x_0 + \max(0, n - y_0)) \\ \leq \max(0, m - x_0) + \max(0, n - y_0)$$

- Optimal bound (rather than minimizing counters and dependencies)
- Scenarios where the invariant generation fails:

Possible research directions

- Nested max

$$\begin{aligned} &\max(0, m - x_0 + \max(0, n - y_0)) \\ &\leq \max(0, m - x_0) + \max(0, n - y_0) \end{aligned}$$

- Optimal bound (rather than minimizing counters and dependencies)
- Scenarios where the invariant generation fails:
    - Invariant generation tool required a global fact

Possible research directions

- Nested max

$$\max(0, m - x_0 + \max(0, n - y_0)) \\ \leq \max(0, m - x_0) + \max(0, n - y_0)$$

- Optimal bound (rather than minimizing counters and dependencies)
- Scenarios where the invariant generation fails:
  - Invariant generation tool required a global fact
  - Linear bounds require path-sensitive invariant generation

Possible research directions

- Nested max

$$\begin{aligned}&\max(0, m - x_0 + \max(0, n - y_0))\\ &\leq \max(0, m - x_0) + \max(0, n - y_0)\end{aligned}$$

- Optimal bound (rather than minimizing counters and dependencies)
- Scenarios where the invariant generation fails:
  - Invariant generation tool required a global fact
  - Linear bounds require path-sensitive invariant generation
- Other types of counters, placement, and dependency

References I

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Science of Computer Programming, 34(3):191–205, 1999.

Sumit Gulwani, Krishna K Mehra, and Trishul Chilimbi.
Speed: precise and efficient static estimation of program computational complexity.
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